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Stress Computations for Nearly Incompressible Materials

Barna A. Szabo, Ivo Babuska, and Bidar K. Chayapathy

April, 1988

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STRESS COMPUTATIONS FOR NEARLY INCOMPRESSIBLE MATERIALS

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In the case of nearly incompressible elastic materials the strain energy, the shear stress, and the difference of normal stresses can be computed accurately by direct methods when the p-version of the finite element method is used. Computation of the sum of the normal stresses requires special procedures. In this paper such procedures are described. Examples are presented.

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KEYWORDS.

Finite Element Analysis, Mechanical Stress, Pressure, Incompressible Materials, Navier-Stokes Equations, Extraction Methods.

TABLE OF CONTENTS.

1. INTRODUCTION.

The problem of computing stresses accurately and reliably in the case of nearly incompressible solids and in the incompressible limit of the compressible Navier-Stokes equations has received a great deal of attention in recent years. Formulations based on the principle of virtual work have been abandoned in favor of mixed and penalty methods in order to avoid locking which occurs when elements of low polynomial degree are used. An overview of the evolution of mixed and penalty methods and a clear exposition of the locking phenomenon is available in [1]. A mixed formulation which has been successfully applied to elastic and inelastic problems is presented in [2]. Mixed formulations have been used successfully in fluid dynamics in conjunction with elements of high polynomial degree [3].

Unlike the formulations presented in [1,2,3], our formulation is based on the principle of virtual work. The solution is obtained in terms of displacements. The sum of normal stresses is recovered by post-solution operations. Our approach to circumventing the locking problem is based on the following:

It has been established theoretically and demonstrated numerically that the error measured in energy norm converges at the same rate independently of Poisson's ratio when the p-version of the finite element method is used [4]. When the material is nearly incompressible (Poisson's ratio is close to 1/2) then direct computation of the sum of the normal stresses from the finite element solution yields very inaccurate results because the volumetric strain is nearly zero in the least squares sense. Small oscillations about the mean value occur, due to errors of approximation in the finite element solution. Direct computation of the sum of the normal stresses involves multiplication of the volumetric strain by a large number, the bulk modulus, which magnifies the errors of approximation [5]. For this reason the sum of normal stresses has to be computed indirectly. On the other hand, the shear stress and the difference of the normal stresses can be computed from the finite element method with good accuracy.

The situation is similar in fluid dynamics when the incompressible limit of the compressible Navier-Stokes equations is approached but the terminology is

Page 1

different: We speak of rate of deformation instead of strain and pressure instead of the sum of normal stresses.

The indirect method described herein utilizes the fact that the sum of the normal stresses satisfies Laplace's equation when the body forces are zero or constant. To solve Laplace's equation it is necessary to specify either the sum of the normal stresses or its derivative with respect to the normal along the boundary of the solution domain. Since we cannot compute these values from the finite element solution directly, we utilize either specified traction values or the equilibrium equation and the shear stresses and the differences of normal stresses computed from the finite element solution.

2. BACKGROUND.

We denote the solution domain by Ω and the thickness of the elastic body by t_z (constant), the displacement vector by $\vec{u}$, and its Cartesian components by $u_x = u_x(x,y)$, $u_y = u_y(x,y)$. The strain components corresponding to $\vec{u}$ are denoted by $\epsilon_x^{(u)}$, $\epsilon_y^{(u)}$, $\gamma_{xy}^{(u)}$. By definition:

$$\epsilon_x^{(u)} \stackrel{\text{def}}{=} \frac{\partial u_x}{\partial x}, \quad \epsilon_y^{(u)} \stackrel{\text{def}}{=} \frac{\partial u_y}{\partial y}, \quad \gamma_{xy}^{(u)} \stackrel{\text{def}}{=} \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}. \tag{1}$$

In the case of *plane strain*, the stress components are related to the strain components by Hooke's law:

$$\sigma_x^{(u)} = \lambda\left(\epsilon_x^{(u)} + \epsilon_y^{(u)}\right) + 2G\epsilon_x^{(u)} \tag{2a}$$

$$\sigma_y^{(u)} = \lambda\left(\epsilon_x^{(u)} + \epsilon_y^{(u)}\right) + 2G\epsilon_y^{(u)} \tag{2b}$$

$$\sigma_z^{(u)} = \lambda\left(\epsilon_x^{(u)} + \epsilon_y^{(u)}\right) \tag{2c}$$

$$\tau_{xy} = G\,\gamma_{xy}^{(u)} \tag{2d}$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad G = \frac{E}{2(1+\nu)} \tag{3}$$

are the Lamé constants. In matrix form (2a,b,c) are written as:

$$\left\{\sigma^{(u)}\right\} = [E]\left\{\epsilon^{(u)}\right\} \tag{4a}$$

where

$$\left\{\sigma^{(u)}\right\} = \left\{\sigma_x^{(u)}\ \sigma_y^{(u)}\ \tau_{xy}^{(u)}\right\}^T, \qquad \left\{\epsilon^{(u)}\right\} = \left\{\epsilon_x^{(u)}\ \epsilon_y^{(u)}\ \gamma_{xy}^{(u)}\right\}^T \tag{4b}$$

and

$$[E] = \begin{bmatrix} \lambda + 2G & \lambda & 0 \\ \lambda & \lambda + 2G & 0 \\ 0 & 0 & G \end{bmatrix}. \tag{4c}$$

$[E]$ is a positive definite matrix and therefore it can be written as:

$$[E] = [T]^T[D][T] \tag{5}$$

where $[D]$ is the diagonal matrix of the eigenvalues of $[E]$ and $[T]$ is the matrix of normalized eigenvectors of $[E]$. Specifically, $[D]$ and $[T]$ are:

$$[D] = \begin{bmatrix} 2(\lambda+G) & 0 & 0 \\ 0 & 2G & 0 \\ 0 & 0 & G \end{bmatrix} \qquad [T] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

Note that $[T] = [T]^T = [T]^{-1}$. The strain energy is:

$$\begin{aligned} U(\vec{u}) &= \frac{1}{2}\iint_\Omega \left(\sigma_x^{(u)}\epsilon_x^{(u)} + \sigma_y^{(u)}\epsilon_y^{(u)} + \tau_{xy}^{(u)}\gamma_{xy}^{(u)}\right) t_z \, dx \, dy \\ &= \frac{1}{2}\iint_\Omega \left\{\epsilon^{(u)}\right\}^T [E] \left\{\epsilon^{(u)}\right\} t_z \, dx \, dy. \end{aligned} \tag{7a}$$

The energy norm of $\vec{u}$ is denoted by $\|\vec{u}\|_E$:

$$\|\vec{u}\|_E \stackrel{\text{def}}{=} \sqrt{U(\vec{u})}. \tag{7b}$$

Using (5) and (6) we have:

$$U(\vec{u}) = \frac{1}{2}\iint_\Omega \left[(\lambda+G)(\epsilon_x^{(u)} + \epsilon_y^{(u)})^2 + G(\epsilon_x^{(u)} - \epsilon_y^{(u)})^2 + G(\gamma_{xy}^{(u)})^2\right] t_z \, dx \, dy. \tag{8}$$

We denote the exact solution (in the virtual work sense) by $\vec{u}_{EX}$. To obtain finite element approximations to $\vec{u}_{EX}$ we construct a mesh Δ by subdividing Ω into triangular and/or quadrilateral elements, some or all of which may have curved sides. The meshes are so constructed that any two elements in the mesh either have a vertex or an entire side in common, or no points in common. We denote the number of elements by $M(\Delta)$. The functions:

$$x = Q_x^{(i)}(\xi,\eta), \quad y = Q_y^{(i)}(\xi,\eta), \quad (\xi,\eta) \in \Omega_{st}^{(t)} \text{ or } (\xi,\eta) \in \Omega_{st}^{(q)}, \quad (i = 1, 2, \ldots M(\Delta)) \tag{9}$$

map a suitably defined standard triangular element $\Omega_{st}^{(t)}$ (resp. standard quadrilateral element $\Omega_{st}^{(q)}$) into the ith triangular (resp. quadrilateral) element. The set of all functions which have finite strain energy on Ω and are polynomials of degree less than or equal p on $\Omega_{st}^{(t)}$, (resp. polynomials of degree less than or equal to p, supplemented by monomial terms of degree $p+1$ on $\Omega_{st}^{(q)}$) is denoted by $S^p(\Omega, \Delta, Q)$

or simply S. Any function in S is continuous on Ω. We define the set of admissible functions to be those functions in S which satisfy the kinematic boundary conditions and denote them by $\tilde{S}$. The dimension of $\tilde{S}$ is called the number of degrees of freedom and is denoted by N.

The finite element solution $\vec{u}_{FE}$ is that function from $\tilde{S}$ which minimizes the strain energy of the error:

$$U\left(\vec{u}_{EX} - \vec{u}_{FE}\right) = \min_{\vec{u}\in\tilde{S}} U\left(\vec{u}_{EX} - \vec{u}\right). \tag{10}$$

Denoting $\vec{e} = \vec{u}_{EX} - \vec{u}_{FE}$, we have

$$U(\vec{e}) = \frac{1}{2}\iint \left[(\lambda + G)\left(\epsilon_x^{(e)} + \epsilon_y^{(e)}\right)^2 + G\left(\epsilon_x^{(e)} - \epsilon_y^{(e)}\right)^2 + G\left(\gamma_{xy}^{(e)}\right)^2\right] t_z\, dx\, dy. \tag{11}$$

The strain energy of the error $U(\vec{e})$ can be progressively reduced by constructing sequences of discretization S_1, S_2, S_3, ... by successive mesh refinement, increase of the polynomial degree of elements, or both. $U(\vec{e})$ approaches zero at a rate which depends on $\vec{u}_{EX}$ and the way in which the finite element spaces $S_1, S_2, S_3 \ldots$ are constructed. Details are available in [6] and the references listed therein. Obtaining a sequence of finite element solutions corresponding to $S_1, S_2, \ldots$ is called extension. If the sequence of finite element spaces is constructed by successive mesh refinement then it is called h-extension, if the mesh is fixed and the polynomial degree is increased then it is called p-extension and if the two are combined then it is called hp-extension. In extensions the number of degrees of freedom is progressively increased.

It has been shown theoretically and demonstrated numerically that in p-extensions $U(\vec{e}) \to 0$ at a rate which is independent of ν, even if ν is very close to its limiting value of $1/2$ [4]. In the case of h-extensions and small p the rate of convergence slows very substantially as $\nu \to 1/2$. The reason for this can be seen from (11): As $\nu \to 1/2$, $\lambda \to \infty$ and the first term of the integrand in (11) can be viewed as a penalty term, i.e. a constraint which requires that $\left(\epsilon_x^{(e)} + \epsilon_y^{(e)}\right)$ be very nearly zero in the least squares sense. One constraint is associated with each element, thereby reducing the number of degrees of freedom by one per element. When h-extension is used and p is low then the number of constraints increases at almost the same rate as N and the rate of convergence slows. In the case of p-extension the number of constraints does not grow, hence nearly incompressible materials can be analyzed without significant loss in effectiveness.

3. COMPUTATION OF STRESSES.

We now define the root-mean-square stress measure $S(\vec{u})$ as follows:

$$S(\vec{u}) \stackrel{\text{def}}{=} \sqrt{\frac{1}{V}\iint_{\Omega}\left[\left(\sigma_x^{(u)}\right)^2 + \left(\sigma_y^{(u)}\right)^2 + \left(\tau_{xy}^{(u)}\right)^2\right] t_z \, dx \, dy} \tag{12}$$

where V is the volume. Using (4a,b,c) and (5) $S^2(\vec{u})$ can be written in terms of the strains as follows:

$$S^2(\vec{u}) = \frac{1}{V}\iint_{\Omega}\left[2(\lambda+G)^2\left(\epsilon_x^{(u)} + \epsilon_y^{(u)}\right)^2 + 2G^2\left(\epsilon_x^{(u)} - \epsilon_y^{(u)}\right)^2 + G^2\left(\gamma_{xy}^{(u)}\right)^2\right] t_z \, dx \, dy. \tag{13}$$

On comparing $S^2(\vec{u})$ with $U(\vec{u})$ in (8) we see that the two expressions are similar. In fact, we can write:

$$\begin{aligned} S_1^2(\vec{e}) &\stackrel{\text{def}}{=} \frac{1}{V}\iint_{\Omega} 2(\lambda+G)^2\left(\epsilon_x^{(e)} + \epsilon_y^{(e)}\right)^2 t_z \, dx \, dy = \frac{1}{2V}\iint_{\Omega}\left(\sigma_x^{(e)} + \sigma_y^{(e)}\right)^2 t_z \, dx \, dy \\ &\leq \frac{4}{V}(\lambda+G)\, U(\vec{e}) \end{aligned} \tag{14a}$$

that is, the square of the error in the sum of the normal stresses integrated over the volume is bounded by the strain energy of the error $U(\vec{e})$ multiplied by a constant which depends on Poisson's ratio and goes to infinity as $\nu \to 1/2$. On the other hand, the error in the differences of normal stresses and shear stress can be made arbitrarily small even if $\nu \to 1/2$:

$$S_2^2(\vec{e}) \stackrel{\text{def}}{=} \frac{1}{V}\iint_{\Omega} 2G^2\left(\epsilon_x^{(e)} - \epsilon_y^{(e)}\right)^2 t_z \, dx \, dy = \frac{1}{2V}\iint_{\Omega}\left(\sigma_x^{(e)} - \sigma_y^{(e)}\right)^2 t_z \, dx \, dy \leq \frac{4}{V} G U(\vec{e}) \tag{14b}$$

and

$$S_3^2(\vec{e}) \stackrel{\text{def}}{=} \frac{1}{V}\iint_{\Omega} G^2\left(\gamma_{xy}^{(e)}\right)^2 t_z \, dx \, dy = \frac{1}{V}\iint_{\Omega}\left(\tau_{xy}^{(e)}\right)^2 t_z \, dx \, dy \leq \frac{2}{V} G U(\vec{e}). \tag{14c}$$

This analysis indicates that we can expect good approximations (in the mean) to $\sigma_x - \sigma_y$ and τ_{xy} from the finite element solution via the constitutive relationships (2a-d) but not to $\sigma_x + \sigma_y$. Therefore we must avoid using the constitutive equations in the computation of the sum of normal stresses. To do this we utilize the fact that when the body forces are zero or constant then the sum of normal stresses satisfies Laplace's equation:

$$\Delta(\sigma_x + \sigma_y) = 0. \tag{15}$$

For details see, for example, [7,8]. Obvious modifications can be made for the case of general body forces. The boundary conditions are established from the equilibrium equations using the fact $\sigma_x^{(uFE)} - \sigma_y^{(uFE)}$ and $\tau_{xy}^{(uFE)}$ can be computed from the finite element solution with sufficient accuracy. We denote typical boundary segments of the solution domain Ω by $\Gamma_{(i)}$, $(i = 1, 2, 3, 4)$ and we denote the positive (outward) unit normal by n, the positive unit tangent by t. Four cases are possible:

(1) σ_n, τ_{nt} are given on Γ_1. Since the sum of normal stresses is invariant with respect to coordinate transformation:

$$\sigma_x + \sigma_y = \sigma_n + \sigma_t = -\left(\sigma_n^{(uFE)} - \sigma_t^{(uFE)}\right) + 2\sigma_n^{(\Gamma_1)} \tag{16}$$

where the superscript (Γ_1) indicates that the function, in this case σ_n, is known from the specified boundary condition.

(2) σ_n, u_t are given on Γ_2. The procedure is the same as in case (1) with Γ_2 replacing Γ_1 in (16). An important special case is the condition of antisymmetry, i. e. $\sigma_n = 0$, $u_t = 0$. In this case $\sigma_x + \sigma_y = 0$.

(3) u_n, τ_{nt} are given on Γ_3. In this case the normal derivative of $\sigma_x + \sigma_y$ is computed from:

$$\frac{\partial(\sigma_x + \sigma_y)}{\partial n} = \frac{\partial(\sigma_n + \sigma_t)}{\partial n} = -\frac{\partial(\sigma_n^{(uFE)} - \sigma_t^{(uFE)})}{\partial n} - 2\frac{\partial \tau_{nt}^{(\Gamma_3)}}{\partial t} + C_3 \tag{17}$$

where we used the equilibrium equation:

$$\frac{\partial \sigma_n}{\partial n} + \frac{\partial \tau_{nt}}{\partial t} = 0. \tag{18}$$

When boundary conditions of type (1) or (2) are imposed on part of the boundary then $C_3 = 0$. When boundary conditions of type (3) are prescribed on the entire boundary then C_3 is a constant determined so that equilibrium is satisfied by the boundary conditions:

$$-\oint \left(\frac{\partial(\sigma_n^{(uFE)} - \sigma_t^{(uFE)})}{\partial n} + 2\frac{\partial \tau_{nt}^{(\Gamma_3)}}{\partial t}\right) ds + C_3 \oint ds = 0. \tag{19}$$

An important special case is the condition of symmetry, i. e. $u_n = 0$, $\tau_{nt} = 0$. In this case $\partial(\sigma_x + \sigma_y)/\partial n = 0$.

(4) u_n, u_t are given on Γ_4:

$$\frac{\partial(\sigma_x+\sigma_y)}{\partial n} = \frac{\partial(\sigma_n+\sigma_t)}{\partial n} = -\frac{\partial(\sigma_n^{(uFE)} - \sigma_t^{(uFE)})}{\partial n} - 2\frac{\partial\tau_{nt}^{(uFE)}}{\partial t} + C_4. \qquad (20)$$

When boundary conditions of type (1) or (2) are imposed on part of the boundary then $C_4 = 0$. When boundary conditions of type (4) are prescribed on the entire boundary then C_4 is a constant determined so that equilibrium is satisfied by the boundary conditions:

$$-\oint\left(\frac{\partial(\sigma_n^{(uFE)} - \sigma_t^{(uFE)})}{\partial n} + 2\frac{\partial\tau_{nt}^{(uFE)}}{\partial t}\right) ds + C_4 \oint ds = 0. \qquad (21)$$

When only boundary conditions of type (3) and (4) are imposed on the entire boundary then $C_3 = C_4 = C$ and C is determined analogously to (19) and (21).

Thus in cases (1) and (2) the sum of the normal stresses is specified on the boundary (Dirichlet condition), in cases (3) and (4) the normal derivative of the sum of the principal stresses is specified (Neumann condition). Either the given boundary condition or the equilibrium equation is used to avoid computation of $(\sigma_x+\sigma_y)$ or its normal derivative from the finite element solution via the constitutive equations. When only cases (3) and (4) are specified along the entire boundary then the pressure can be determined only up to an arbitrary constant.

The stresses computed from the finite element solution are not continuous but the sum of the normal stresses specified on the boundary in cases (1) and (2) has to be continuous. For this reason some averaging of the values at the nodes is necessary.

3.1 Points of stress singularity.

When one or more points on the boundary are points of stress singularity then the procedure just described has to be modified as follows: In the neighborhood of singular points the exact solution can be written in the form:

$$\vec{u}_{EX} = \sum_{i=1}^{\infty} A_i r^{\lambda_i} \vec{\Psi}_i(t), \qquad r < r_0 \qquad (22)$$

where r, θ are polar coordinates centered on the singular point; the coefficients A_i are called generalized stress intensity factors; λ_i and $\psi_i(\theta)$ are determined from the conditions that $\vec{u}_{EX}$ must satisfy the equations of equilibrium and the boundary conditions in the neighborhood of the singular points; $r_0 > 0$ is the radius of convergence of the infinite series (22).

The coefficients A_i can be computed from the finite element solution by a method so that the accuracy of A_i depends only on the accuracy of the finite element solution $\vec{u}_{FE}$ measured in energy norm and the accuracy of the finite element solution to an auxiliary problem $\vec{w}_{FE}$, also measured in energy norm, which differs from the original problem only in loading. Thus:

$$|(A_i)_{EX} - (A_i)_{FE}| \leq \|\vec{u}_{EX} - \vec{u}_{FE}\|_E \|\vec{w}_{EX} - \vec{w}_{FE}\|_E \tag{23}$$

The function $\vec{w}_{FE}$ is not actually computed, it serves theoretical purposes only. In general the errors $\|\vec{u}_{EX} - \vec{u}_{FE}\|_E$ and $\|\vec{w}_{EX} - \vec{w}_{FE}\|_E$ are of comparable magnitude, hence the error in the computed values of A_i is comparable with the error in strain energy. For details see references [9-12]. Since the accuracy of A_i depends only on the accuracy of solutions measured in energy norm which, in the case of p-extensions, is not sensitive to Poisson's ratio, it is possible to compute A_i accurately independently of Poisson's ratio.

Once A_i are known, the loading corresponding to the stress singular term can be subtracted from the applied loading and the problem solved as a smooth problem. Alternatively a small neighborhood of the singular point can be "removed" from the domain and either the displacements or tractions corresponding to (22) imposed on the boundary which defines the small neighborhood. An example is given in Section 5.

4. EXAMPLE: RIGID CIRCULAR INCLUSION IN AN ELASTIC PLATE.

We will investigate the case of a rigid circular inclusion in an infinite plate, subjected to unidirectional tension. Plane strain conditions are assumed. The notation is shown in Fig. 1. This example is typical of cases where all derivatives of the exact solution are continuous and bounded on the entire solution domain, including the boundary of the solution domain, but one or more singular points lie outside of the solution domain.

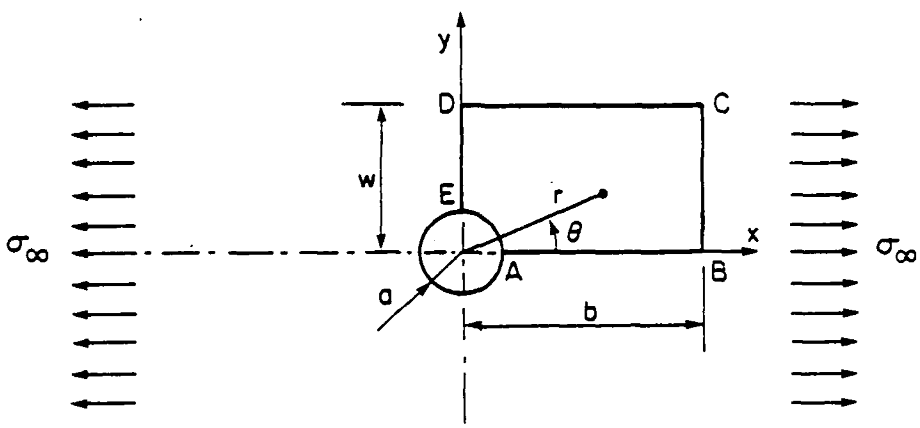

Fig. 1. Notation.

4.1. The exact solution.

In this case the exact solution is available [8]. The exact displacement components are:

$$u_r = \frac{\sigma_\infty}{8Gr}\left\{(\kappa-1)\,r^2 + 2\gamma a^2 + \left[\beta(\kappa+1)\,a^2 + 2r^2 + \frac{2\delta a^4}{r^2}\right]\cos 2\theta\right\} \tag{24a}$$

$$u_\theta = -\frac{\sigma_\infty}{8Gr}\left[\beta(\kappa-1)\,a^2 + 2r^2 - \frac{2\delta a^4}{r^2}\right]\sin 2\theta \tag{24b}$$

and the exact stress components are:

$$\sigma_r = \frac{\sigma_\infty}{2}\left[1 - \frac{\gamma a^2}{r^2} + \left(1 - \frac{2\beta a^2}{r^2} - \frac{3\delta a^4}{r^4}\right)\cos 2\theta\right] \tag{25a}$$

$$\sigma_\theta = \frac{\sigma_\infty}{2}\left[1 + \frac{\gamma a^2}{r^2} - \left(1 - \frac{3\delta a^4}{r^4}\right)\cos 2\theta\right] \tag{25b}$$

$$\tau_{r\theta} = -\frac{\sigma_\infty}{2}\left(1 + \frac{\beta a^2}{r^2} + \frac{3\delta a^4}{r^4}\right)\sin 2\theta \tag{25c}$$

where κ, β, γ, δ are constants which depend on Poisson's ratio ν only. In the case of *plane strain*:

$$\kappa = 3 - 4\nu; \quad \beta = -\frac{2}{3-4\nu}; \quad \gamma = -\frac{2-4\nu}{2}; \quad \delta = \frac{1}{3-4\nu}. \tag{26}$$

The center of the rigid circular inclusion is a singular point.

4.2. Finite element solutions.

Finite element solutions were obtained by means of the computer program PROBE [13]. The solution domain is defined by $a = 1$, $w = b = 4$ in Fig. 1. The domain and two finite element meshes, a four-element mesh and an eight-element mesh are shown in Fig. 2.

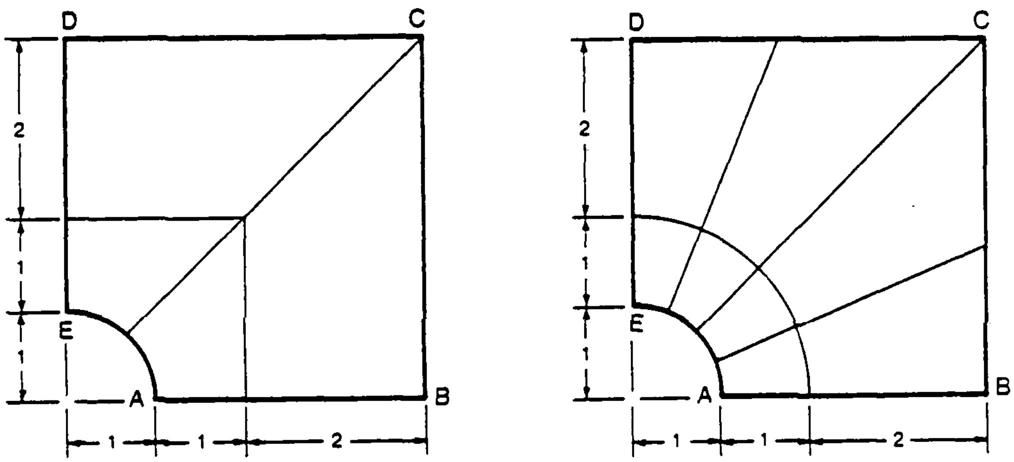

Fig. 2. Solution domain and finite element meshes.

The boundary conditions are as follows: Along the circular arc AE both displacement components are zero. Along the symmetry lines AB and DE the normal displacement component and the shear stress are zero. Along boundaries BC and CD the tractions computed from (25.a,b,c) are imposed. The corresponding load vectors were computed by numerical quadrature using twelve Gauss points per element side.

4.2.1. Four-element mesh, $0 \le \nu \le 0.4999999$.

The exact value of the strain energy for $\nu = 0.4999$, computed from (24a,b) and (25a,b,c), is $5.79590408\ \sigma_\infty^2 a^2 t_z/E$. The number of degrees of freedom N; the strain energy values computed from the finite element solutions corresponding to p ranging from 1 to 8; the estimated algebraic rate of convergence 2β, and the estimated and true relative errors in energy norm are given in Table 1. The estimated algebraic rate of convergence 2β is an estimate of the absolute value of the slope of the $\log U(\vec{e})$ vs. $\log N$ curve. The estimated relative errors in energy norm, defined by

$$(e_r)_E \stackrel{\text{def}}{=} 100 \frac{\|\vec{u}_{EX} - \vec{u}_{FE}\|_E}{\|\vec{u}_{EX}\|_E} \tag{27}$$

were computed by the method described in [13,14]. We see that the relative error in energy norm is under one percent at $p = 8$.

Table 1. Convergence of the strain energy.
Estimated and true relative errors in energy norm.
Four-element mesh, $\nu = 0.4999$.

p	N	$\frac{U(\vec{u}_{FE})E}{\sigma_\infty^2 a^2 t_z}$	Est.'d 2β	Est.'d $(e_r)_E$ (%)	True $(e_r)_E$ (%)
1	8	0.35682	–	96.87	96.87
2	24	3.15500	0.66	67.50	67.50
3	40	5.16100	2.79	33.10	33.10
4	64	5.66227	3.31	15.19	15.18
5	96	5.75195	2.74	8.72	8.71
6	136	5.78552	4.12	4.26	4.23
7	184	5.79388	5.27	1.92	1.87
8	240	5.79550	5.27	0.95	0.84
∞	∞	5.79590	∞	0	0

The relationship between relative error in energy norm and the polynomial degree of approximation for other values of Poisson's ratio are shown in Fig. 3. Note that the relative error in energy norm is plotted on a logarithmic scale against the *square root* of the number of degrees of freedom N. The reason for this is that when the exact solution is analytic on the solution domain, including its boundary, then the rate of convergence of p-extensions is exponential:

$$(e_r)_E \leq \frac{k}{\exp(\gamma\sqrt{N})} \tag{28}$$

where k and γ are positive constants, [6]. Constant k depends on Poisson's ratio but γ does not. This is demonstrated by the numerical results summarized in Fig.

3 which show that the relative error in energy norm depends on ν but its rate of change with respect to $\sqrt{N}$ is constant.

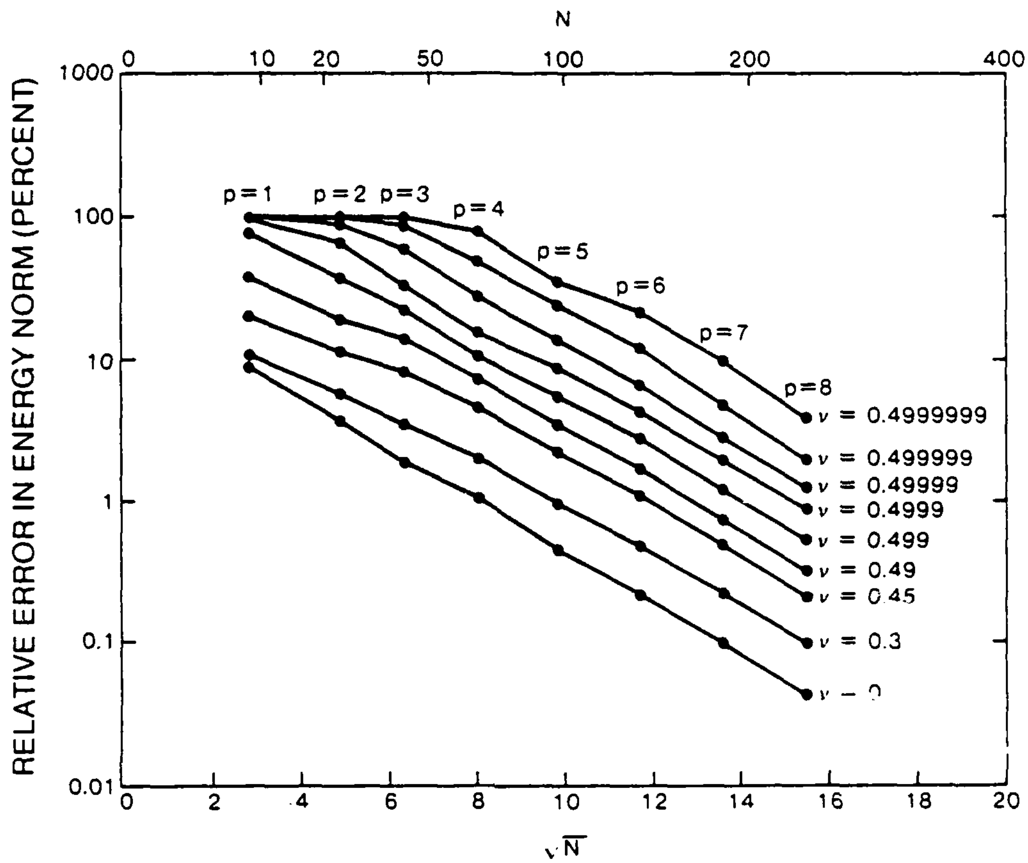

Fig. 3. True relative error in energy norm vs. $\sqrt{N}$ for various Poisson's ratios. Four-element mesh.

4.2.2. Eight-element mesh, $\nu = 0.4999999$.

Finite element solutions were also obtained with the eight-element mesh shown in Fig. 2b for $\nu = 0.4999999$. The exact value of the strain energy in this case is is $5.79517057\ \sigma_\infty^2 a^2 t_z/E$. Convergence of the strain energy for p ranging from 1 to 8 and the estimated and true relative errors in energy norm are shown in Table 2. Once again the relative error in energy norm is under one percent at $p = 8$.

4.3. Computation of the sum of the normal stresses.

The sum of the normal stresses was computed with the four-element mesh

Table 2. Convergence of the strain energy.
Estimated and true relative errors in energy norm.
Eight-element mesh, $\nu = 0.4999999$.

p	N	$\frac{U(\vec{u}_{FE})E}{\sigma_\infty^2 a^2 t_z}$	Est.'d 2β	Est.'d $(e_r)_E$ (%)	True $(e_r)_E$ (%)
1	16	0.00029	0.00	100.00	100.00
2	48	0.03364	0.01	99.71	99.71
3	80	2.69231	1.21	73.17	73.17
4	128	5.52142	5.17	21.74	21.73
5	192	5.71015	2.88	12.12	12.11
6	272	5.78156	5.25	4.85	4.85
7	368	5.79393	7.79	1.49	1.46
8	480	5.79506	7.79	0.53	0.44
∞	∞	5.79517	∞	0	0

shown in Fig. 2a for $\nu = 0.4999$ and with the eight-element mesh shown in Fig. 2b for $\nu = 0.4999999$.

Along the circular segment AE the normal derivative of the sum of the normal stresses was computed from (20). Along the symmetry lines AB, DE the natural boundary condition $\partial(\sigma_x + \sigma_y)/\partial n = 0$ was imposed and along boundaries BC, CD $\sigma_x + \sigma_y$ was computed from (16). In order to utilize existing capabilities of the computer program PROBE as much as possible, the normal derivative was computed numerically at nine equally spaced points along each of the two curved element boundaries. The normal derivatives were then determined at 12 Gauss points by interpolation. The load vector terms were computed by Gaussian quadrature, using 12 quadrature points. With appropriate modifications in the program, transfer of boundary conditions from the solution of the elasticity problem to the Laplace problem can be automated so that no loss in accuracy is incurred in the transfer itself.

The essential boundary conditions were handled by computing $(\sigma_x + \sigma_y)$ at ten equally spaced points along each element boundary on AB and BC, including the vertices. At the vertices the computed values were averaged. For example, in the case of the four-element mesh ($\nu = 0.4999$) and $p = 8$, at vertex C:

$$(\sigma_x + \sigma_y)_C = \frac{1}{2}(0.9980777 + 1.0019223)\,\sigma_\infty = 1.0000000\,\sigma_\infty \tag{29}$$

which agrees to at least eight significant digits with the exact value of $(\sigma_x + \sigma_y)$. Least squares approximation was used to obtain coefficients of the basis functions

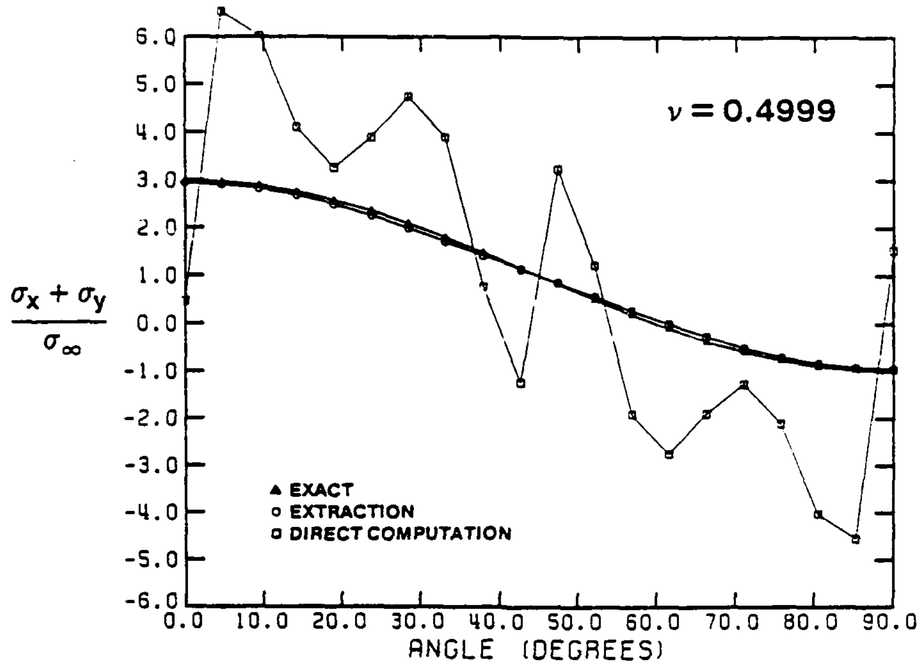

Fig. 4a. The sum of normal stresses along the rigid inclusion. Four-element mesh, $p = 8$, $\nu = 0.4999$.

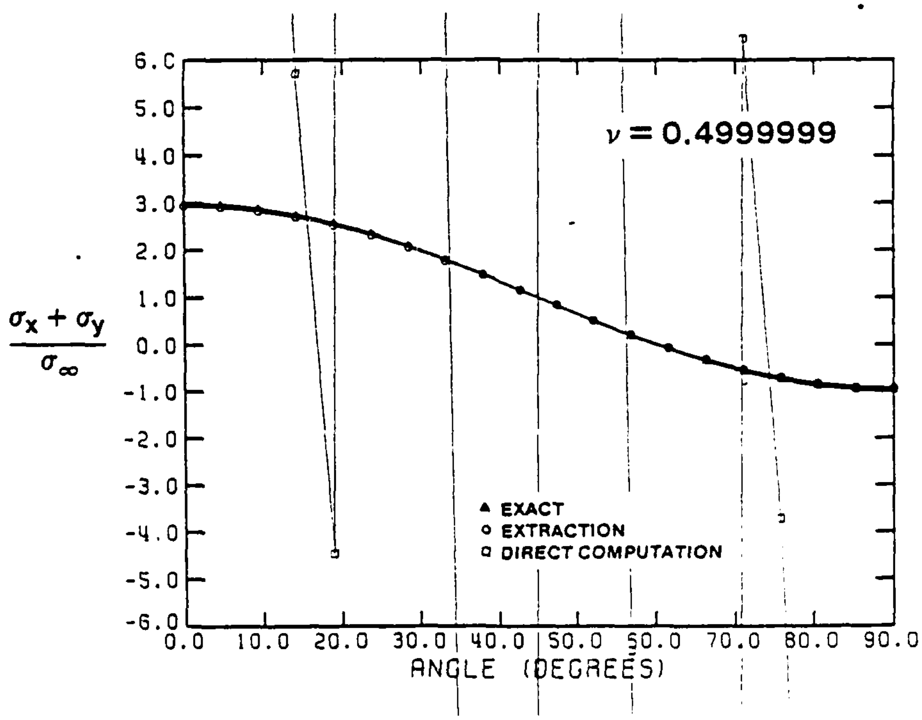

Fig. 4b. The sum of normal stresses along the rigid inclusion. Eight-element mesh, $p = 8$, $\nu = 0.4999999$.

along the element boundaries with the restriction that the values at the vertices were fitted exactly.

The sum of the normal stresses along the rigid inclusion, computed directly from the finite element solution, and by the extraction procedure described in this paper, are plotted along with their exact values in Fig. 4a and Fig. 4b. The polynomial degree 8 was used in solving both the elasticity problem and the Laplace problem. The estimated and true relative errors in energy norm for the elasticity problem are in Tables 1 and 2. In the case of the Laplace problem the estimated relative errors in energy norm were as follows: Four-element mesh, $p = 8$, $N = 136$, $(e_r)_E = 0.10\%$. Eight-element mesh, $p = 8$, $N = 2{:}3$, $(e_r)_E = 0.04\%$.

The results shown in Figures 4a,b are indicative of the quality of approximation obtained for the entire domain. A contour plot for the sum of normal stresses, computed with the eight-element mesh, $\nu = 0.4999999$, is shown in Fig. 5.

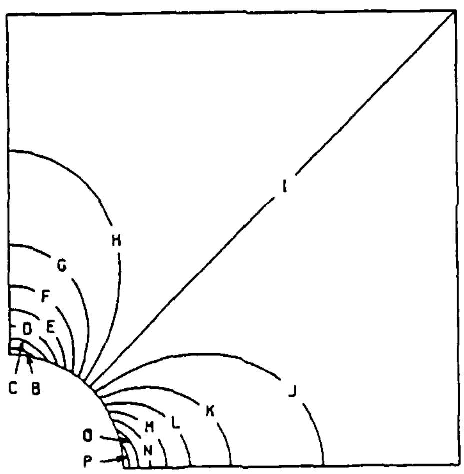

Fig. 5. Contour plot of the sum of normal stresses.
Eight-element mesh, $p = 8$, $\nu = 0.4999999$. Contour interval: $0.25\,\sigma_\infty$.
$B:\ -0.75\,\sigma_\infty$, $E:\ 0$, $I:\ 1.00\,\sigma_\infty$, $M:\ 2.00\,\sigma_\infty$, $P:\ 2.75\,\sigma_\infty$.

Equations (14b,c) show that $\sigma_x - \sigma_y$ and τ_{xy} converge in the least squares sense at the same rate as the error in energy norm does, which is exponential. Exponential convergence can be proven for pointwise values of $\sigma_x - \sigma_y$, τ_{xy} and their derivatives also, but the proof is beyond the scope of this paper. This example demonstrates that control of errors in boundary data does not require large computational effort.

5. EXAMPLE: L-SHAPED DOMAIN.

This example is typical of cases where, due to reentrant corners, sudden changes in boundary conditions, material properties or loading, one or more singular points lie on the boundary of the solution domain.

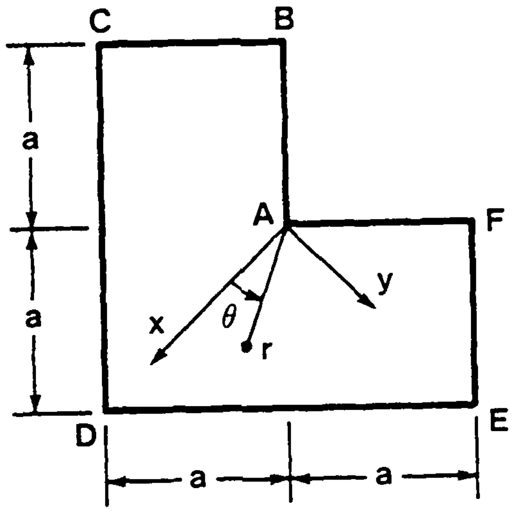

Fig. 6. L-shaped domain. Notation.

5.1. The exact solution.

The L-shaped plane elastic body, shown in Fig. 6, is loaded by tractions which correspond to the following exact displacement field:

$$u_x = \frac{A_1}{2G} r^{\lambda_1} \left[(\kappa - Q_1(\lambda_1 + 1)) \cos \lambda_1 \theta - \lambda_1 \cos(\lambda_1 - 2)\theta \right] \tag{30a}$$

$$u_y = \frac{A_1}{2G} r^{\lambda_1} \left[(\kappa + Q_1(\lambda_1 + 1)) \sin \lambda_1 \theta + \lambda_1 \sin(\lambda_1 - 2)\theta \right] \tag{30b}$$

where A_1 is a constant, analogous to the mode 1 stress intensity factor in linear elastic fracture mechanics; $\lambda_1 = 0.544483737$, $Q_1 = 0.543075579$ are constants determined so that the solution satisfies the equations of equilibrium and the stress free boundary conditions on the reentrant edges. $G = E/2(1+\nu)$ is the modulus of rigidity; κ is a constant which depends on Poisson's ratio and whether plane stress or plane strain conditions are assumed. In this example plane strain conditions are assumed hence $\kappa = 3 - 4\nu$. The stress components are:

$$\sigma_x = A_1 \lambda_1 r^{\lambda_1 - 1} [(2 - Q_1(\lambda_1 + 1)) \cos(\lambda_1 - 1)\theta - (\lambda_1 - 1)\cos(\lambda_1 - 3)\theta] \tag{31a}$$

$$\sigma_y = A_1 \lambda_1 r^{\lambda_1 - 1} [(2 + Q_1(\lambda_1 + 1)) \cos(\lambda_1 - 1)\theta + (\lambda_1 - 1)\cos(\lambda_1 - 3)\theta] \tag{31b}$$

$$\tau_{xy} = A_1 \lambda_1 r^{\lambda_1 - 1} [(\lambda_1 - 1)\sin(\lambda_1 - 3)\theta + Q_1(\lambda_1 + 1)\sin(\lambda_1 - 1)\theta]. \tag{31c}$$

The boundary conditions are as follows: Along the reentrant edges AB and FA the normal and shear stresses are zero; along edges BC, CD, DE and EF the normal and shear stresses corresponding to (28a,b,c) are prescribed. These tractions satisfy equilibrium hence only rigid body constraints have to be applied. The exact value of the strain energy is:

$$U(\vec{u}_{EX}) = 2.35967\, A_1^2 a^{2\lambda_1} t_z / E. \tag{32}$$

Additional information concerning this model problem is available in [12,14].

5.2. Finite element solutions.

In this example the eighteen-element mesh shown in Fig. 7 and $\nu = 0.4999999$ was used. The coefficient A_1 was determined by the cutoff function method described in [12,13]. The results of finite element solutions corresponding to p ranging from 1 to 8 are given in Table 3.

Table 3. Convergence of the strain energy and the computed value of the generalized stress intensity factor A_1. Eighteen-element mesh, $\nu = 0.4999999$.

p	N	$\frac{U(\vec{u}_{FE})}{U(\vec{u}_{EX})}$	$\frac{(A_1)_{FE}}{(A_1)_{EX}}$
1	41	0.00184	3.06910
2	119	0.42400	25.27936
3	209	0.84901	2.48314
4	335	0.94668	1.30447
5	497	0.99538	1.02696
6	695	0.99761	1.01301
7	929	0.99949	1.00296
8	1199	0.99974	1.00105

These results show that the errors in strain energy and A_1 are of comparable magnitude and A_1 converges at approximately the same rate as the strain energy

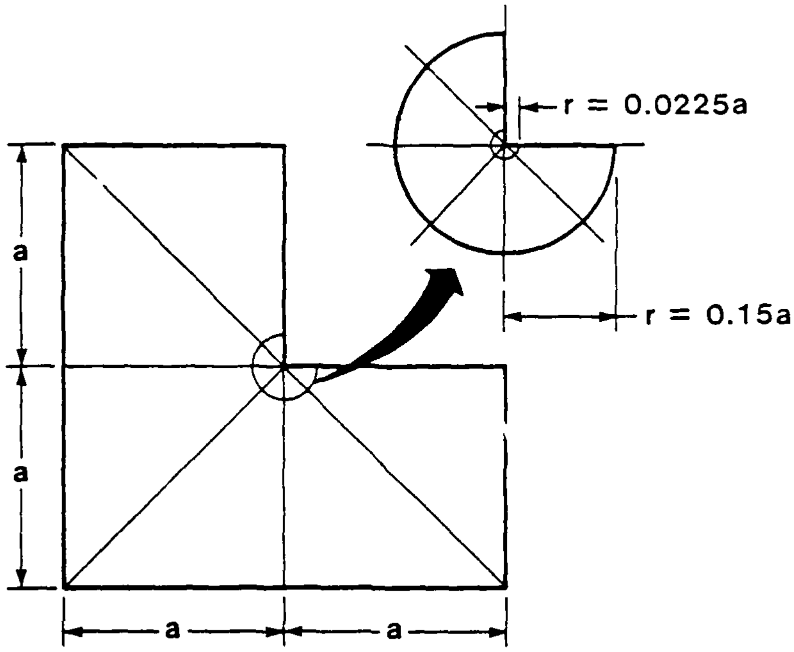

Fig. 7. L-shaped domain. Eighteen-element mesh.

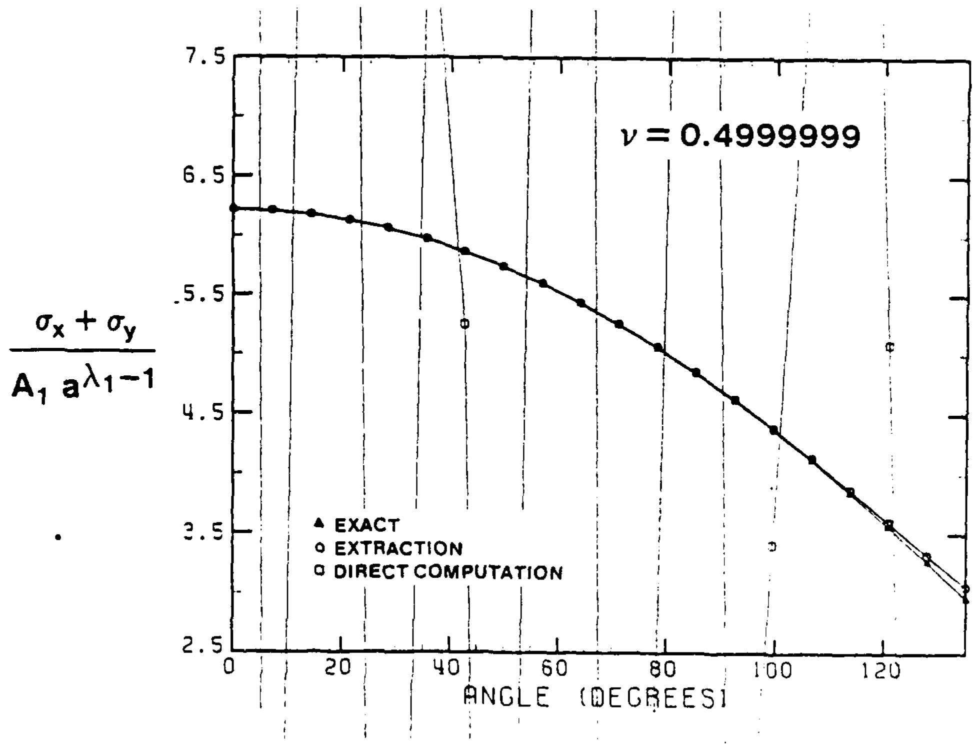

Fig. 8. L-shaped domain. Sum of normal stresses at $r = 0.1a$ $(p = 8)$.

does. This is consistent with the theoretical estimate for extraction procedures (23). In general, data computed by extraction procedures exhibit *superconvergence.*

5.3. Computation of the sum of normal stresses.

In the close neighborhood of the singular point the exact solution is approximated by the expression (22) with A_i computed by the cutoff function method described in [12]. Thus the error in the displacements and stresses, and any other quantity computed from (22), is controlled by the error in the coefficients A_i. In this model problem the exact solution is the first term in (22) hence we need not be concerned with the question of where to truncate the infinite series (22). We note, however, that the coefficient of the second term, corresponding to the first antisymmetric mode of deformation ($\lambda_2 = 0.90853$), computed from the finite element solution of $p = 8$ by the cutoff function method, is $2.38 \times 10^{-8}\,(A_1)_{EX}$. From Table 3 we see that A_1 is accurate to 0.1 percent, hence all stress components are also accurate to 0.1 percent.

In order to simulate the case where the stresses are computed from (22) only in the immediate neighborhood of the singular point but elsewhere the method described in Section 4 is used, we removed from the solution domain the circular sector of radius $0.0225\,a$, defined by the inner circle in Fig. 7. We computed the sum of normal stresses corresponding to (22) along the boundary of this sector. We imposed the symmetry boundary condition along the x-axis (see Fig. 6) and along the other boundary segments we computed the sum of normal stresses from (16). The computed sum of normal stresses along a circle of radius $0.1\,a$ is shown in Fig. 8. We see that the accuracy is about the same as in the case of smooth solutions discussed in Section 4.

The results of this example are typical of the performance of hp-extensions where the error in energy norm is of the order: $\exp(-\gamma N^{1/3})$, $\gamma > 0$. Once again, it can be proven that convergence of $\sigma_x - \sigma_y$, τ_{xy} and their derivatives, computed at points chosen independently of the mesh refinement, has the same exponential character. This explains the high accuracy of the results in this example.

6. NOTE ON THE INCOMPRESSIBLE LIMIT OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS.

The mathematical formulation of compressible Navier-Stokes equations is very different from the mathematical formulation of the linear elasticity problem. Nevertheless, in the case of viscous fluids which satisfy certain restrictions on the first and second viscosity coefficients, the limiting case with respect to the Mach number approaching zero is exactly analogous to the limiting case of incompressible elasticity with respect to Poisson's ratio approaching 1/2. Precise statement of the necessary conditions and rigorous proof is available in [15].

7. SUMMARY AND CONCLUSIONS.

We have described a method for the computation of stress components in the case of nearly incompressible isotropic elastic materials. We have shown that in those problems where the exact solution is smooth the shear stresses and the differences of normal stresses can be computed directly from the finite element solution, provided that the error measured in energy norm is sufficiently small. For this class of problems p-extensions provide efficient means for reducing the errors of approximation and the performance of p-extensions is not sensitive to Poisson's ratio. Thus yield criteria which are independent of the sum of normal stresses, such as the maximum shear stress criterion, can be applied directly, independently of Poisson's ratio.

When Poisson's ratio is close to 1/2 then the sum of the normal stresses has to be determined indirectly. When the body forces are zero or constant then this involves solution of the Laplace equation, the boundary conditions of which are determined from the finite element solution of the elasticity problem and the specified boundary tractions.

In those problems where one or more points of stress singularity occur on the boundary of the solution domain the procedure is similar to the case of smooth solution but in the neighborhood of singular points the approximation is in terms of the coefficients of an asymptotic expansion which can be computed by methods described in [7-10]. Important advantages of these methods are that they are very efficient and their accuracy is not sensitive to Poisson's ratio.

In the method described herein the solution for the sum of normal stresses is decoupled from the solution of the displacement components in the sense that first the displacement components are approximated, independently of the sum of the normal stresses, then the sum of normal stresses is approximated in a separate solution step. In mixed formulations such decoupling generally does not occur.

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